Eur. Phys. J. C 39, 511–518 (2005)
Digital Object Identifier (DOI) 10.1140/epjc/s2004-02093-x

THE EUROPEAN
PHYSICAL JOURNAL C

Non-equivalence between Heisenberg XXZ spin chain and Thirring model

T. Fujita[a], T. Kobayashi[b], M. Hiramoto[c], H. Takahashi[d]

Department of Physics, Faculty of Science and Technology, Nihon University, Tokyo, Japan

Received: 28 October 2004 /
Published online: 21 January 2005 –

Abstract. The Bethe ansatz equations for the spin 1/2 Heisenberg XXZ spin chain are numerically solved, and the energy eigenvalues are determined for the antiferromagnetic case. We examine the relation between the XXZ spin chain and the massless Thirring model, and show that the spectrum of the XXZ spin chain has a gapless excitation while the regularized Thirring model calculated with the Bogoliubov transformation method has a finite gap. This finite gap spectrum is also confirmed by the Bethe ansatz solution of the massless Thirring model.

PACS. 10.Kk, 03.70.+k, 11.30.-j, 11.30.Rd

1 Introduction

Symmetry breaking has been discussed quite extensively in varieties of field theories [1, 2]. It is believed that spontaneous symmetry breaking should accompany a massless boson, and since there should not physically exist a massless boson in two dimensional field theory due to the infra-red singularity of the massless boson propagator, symmetry breaking should not occur in a two dimensional field theory model [3].

However, fermion field theory models are quite different in that bosons must be dynamically constructed by fermions and antifermions. For the fermion field theory models, the Goldstone boson has been known only for the current current interaction model of Nambu and Jona-Lasinio (NJL model) [4]. However, recent careful studies prove that there is no massless boson in the NJL model after chiral symmetry breaking [5,6]. The physics of chiral symmetry breaking is rather simple. The chiral symmetry which is possessed in the NJL Lagrangian with massless fermion is broken in the new vacuum since the new vacuum is lower than the trivial one. If one employs the Bogoliubov transformation method as done by Nambu and Jona-Lasinio, then one finds that the *originally* massless fermion acquires a finite mass, and becomes a massive NJL model which predicts always a massive boson, and the boson can never become massless since the induced fermion mass can never be set to zero.

However, Nambu and other people obtained a massless boson. The reason is simple: they calculated the boson mass by summing up the one loop Feynman diagrams based on the perturbative vacuum state (false vacuum state). This is, of course, a wrong procedure and therefore the massless boson they obtained did not depend on the strength of the coupling constant [7].

However, if one carries out the calculations of the boson mass by the formulation based on the new vacuum state (symmetry broken vacuum state), then one obtains a massive boson, depending on the strength of the coupling constant.

In the same way, the chiral symmetry in the massless Thirring model is broken in the new vacuum [8], but there appears no massless boson [5,6]. Since the massless Thirring model is believed to be equivalent to the spin 1/2 Heisenberg XXZ model at the continuum limit [9, 10], it should be interesting to compare the results of the energy eigenvalues of the two models at the continuum limit.

In this paper, we solve numerically the Bethe ansatz equations for the spin 1/2 Heisenberg XXZ model [11, 12] and obtain the energy eigenvalues. The XXZ spin chain has gapless excitations and is consistent with previous results. On the other hand, we calculate the energy eigenvalues of the regularized Thirring model in the Bogoliubov transformation method [5,13] and find that the regularized Thirring model has a finite gap and a massive boson state. Therefore, it is clear that the Heisenberg XXZ model and the massless Thirring model do not agree with each other since the spectra are different from each other.

Since the Bogoliubov transformation is not necessarily exact, we have also carried out the calculation of the Bethe ansatz solutions for the Thirring model [14]. Here, we confirm that the vacuum has a chiral symmetry broken state and is lower than the symmetric vacuum, and there is indeed a finite gap in the excitation spectrum [15]. How-

[a] e-mail: fffujita@phys.cst.nihon-u.ac.jp
[b] e-mail: tkoba@phys.cst.nihon-u.ac.jp
[c] e-mail: hiramoto@th.phys.titech.ac.jp
[d] e-mail: htaka@phys.ge.cst.nihon-u.ac.jp

ever, we cannot find any boson state in the Bethe ansatz solution of the Thirring model. This indicates that the Bogoliubov transformation method of the Thirring model is indeed not exact and tends to overestimate the attractions between fermions and antifermions. However, it can describe the chiral symmetry breaking phenomena. Also, the Bogoliubov transformation method can describe the boson spectrum of QED_2 [16] and QCD_2 [17] reasonably well.

Here, we discuss the physics behind the difference between the two models. The equivalence between the spin 1/2 Heisenberg XYZ model and the massive Thirring model is well established [9]. But the massless limit in the massive Thirring model is a singular point and should not be taken naively. The massless Thirring has two types of vacuum, one of which corresponds to the trivial vacuum with the chiral symmetry preserved, and the other one is a true vacuum with breaking of the chiral symmetry. One sees that the true vacuum state is lower than the trivial one, and therefore the true vacuum is physically realized. From the present analysis, we show that the XXZ spin chain cannot be reduced to the Thirring model with the true vacuum even though one may mathematically obtain the Thirring Lagrangian from the XXZ spin chain.

The proof for the equivalence between the XXZ spin chain and the Thirring model is based on the naive continuum limit of the XXZ model. However, the XXZ model has only one scale, and therefore, the physical meaning of the continuum limit is not clear. All the physical observables are measured by the lattice constant a, and thus in order that the *very small* a makes sense, one should compare a with another scale quantity. In this respect, the continuum limit of the XYZ can be well defined, but the XXZ model should keep the lattice constant a finite. Even if one says that one could derive the field theory model at the continuum limit (small lattice constant a), all the observables of this field theory model should be measured by the lattice constant a. In the Thirring Lagrangian derived mathematically from the XXZ spin chain, there is no scale parameter corresponding to the lattice constant a, and physically, this indicates that the XXZ spin chain and the Thirring model must be different from each other.

This paper is organized in the following way. In the next section, we briefly explain the Bethe ansatz solutions for the XXZ spin chain. Section 3 treats the massless Thirring model in the Bogoliubov transformation method, and the energy eigenvalues of the vacuum and the excited states with the chiral symmetry breaking are discussed. In Sect. 4, we present the Bethe ansatz solutions of the massless Thirring model and show that the symmetry broken vacuum of the Thirring model is indeed the lowest state. In Sect. 5, we discuss the equivalence between the XXZ spin chain and the massless Thirring models. Section 6 summarizes what we have learned here.

2 Heisenberg XXZ model

Here, we briefly describe the Heisenberg XXZ model. The XXZ model has the following Hamiltonian [11,12]:

$$H = J \sum_{i=1}^{N} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \varDelta S_i^z S_{i+1}^z \right) , \tag{2.1}$$

where S_i^a is a spin operator at the site i. J and $\varDelta$ denote the coupling constant and the anisotropy parameter, respectively, and N is the site number. The periodicity $S_{N+i} = S_i$ is assumed. This Hamiltonian can be numerically solved by exact diagonalization. However, if one wants to discuss the excitation spectrum, then one has to have a site number N larger than $N = 1000$ or so [18]. This is practically impossible.

Fortunately, this model is solved by the Bethe ansatz technique, and the Hamiltonian can be diagonalized by the superposition of the wave functions $\phi(z_{n_1}, \ldots, z_{n_m})$ for the m down spin case as follows:

$$\Psi = \sum_P A(n_1, \ldots, n_m) \phi(z_{n_1}, \ldots, z_{n_m}) , \tag{2.2}$$

where P means all possible permutations of the $n_1, \ldots, n_m$. Further, the coefficient $A(n_1, \ldots, n_m)$ is assumed to be of the following shape:

$$\begin{aligned} &A(n_1, \ldots, n_m) \\ &= \sum_{P_\mu} \sum_P \exp\left(\mathrm{i} \sum_j^m k_{P_j} n_{\mu_j} + \frac{1}{2} \sum_{j<\ell} \varphi_{P_j P_\ell} \right) , \end{aligned} \tag{2.3}$$

where the k_i denote the pseudo-momentum of the down spin site. From the periodic boundary conditions, we obtain the following equations:

$$N k_j = 2\pi \lambda_j + \sum_\ell \varphi_{j\ell} , \tag{2.4}$$

where λ_i are integers running between 0 and $N-1$ with the condition of $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_m$. The equation for $\varphi_{j\ell}$ becomes

$$\cot \frac{\varphi_{j\ell}}{2} = \frac{\varDelta \sin\left(\frac{k_j - k_\ell}{2}\right)}{\cos\left(\frac{k_j + k_\ell}{2}\right) - \varDelta \cos\left(\frac{k_j - k_\ell}{2}\right)} . \tag{2.5}$$

In this case, we can express the energy eigenvalue E as

$$E = \left(\frac{1}{4} N - m \right) \varDelta + \sum_{j=1}^m \cos k_j . \tag{2.6}$$

The Bethe ansatz equations (2.5) can be numerically solved by the new iteration method which is developed in [19,20].

Here, we should write the translation of the coupling constants between the spin chain and the Thirring model [9], and the Thirring coupling constant g is related to the $\varDelta$ by

$$g = \frac{4\pi\varDelta}{2\pi - \varDelta} . \tag{2.7}$$

It should be noted that the correspondence between the two models is only meaningful for the condition

$$\varDelta \le \frac{2}{5}\pi , \tag{2.8}$$

since g must be smaller than π [9]. The vacuum state of the field theory corresponds to the state of $S_z = 0$, which is just the antiferromagnetic state. In Sect. 4, the numerical results of the excitation spectrum will be discussed.

Further, we briefly describe the procedure commonly employed to obtain the Thirring model Lagrangian from (2.1) [10]. By the Jordan–Wigner transformation, one can rewrite the Hamiltonian of (2.1) in terms of the spinless lattice fermion:

$$H = J\sum_{i=1}^{N}\left[\frac{1}{2}\left(\psi_i^\dagger\psi_{i+1} + \text{h.c.}\right) + \Delta\left(\psi_i^\dagger\psi_i - \frac{1}{2}\right)\left(\psi_{i+1}^\dagger\psi_{i+1} - \frac{1}{2}\right)\right]. \quad (2.9)$$

This Hamiltonian can be reduced to the massless Thirring model Lagrangian below when one takes naively the continuum limit [10].

3 Thirring model

The massless Thirring model is described by the following Lagrangian density [21]:

$$\mathcal{L} = \mathrm{i}\bar{\psi}\gamma_\mu\partial^\mu\psi - \frac{g}{2}j^\mu j_\mu\,, \quad (3.1)$$

where the fermion current j_μ is given by $j_\mu =: \bar{\psi}\gamma_\mu\psi :$.

This model is studied by the Bogoliubov transformation, and it is found that the vacuum has a chiral symmetry broken phase [5,6,13]. Therefore, the fermion has the following mass:

$$\mathcal{M} = \frac{\Lambda}{\sinh(\pi/g)}\,. \quad (3.2)$$

The vacuum energy E_{vac} as measured from the trivial vacuum ($E_{\text{vac}}^{\text{triv.}} = 0$) is given by

$$E_{\text{vac}} = -\frac{L}{2\pi}\frac{\Lambda^2}{\sinh(\pi/g)}\mathrm{e}^{-\frac{\pi}{g}}\,, \quad (3.3)$$

where Λ and L denote the cutoff momentum and box length in this model, respectively, and all of the physical quantities must be measured by the Λ.

Now, in order to compare directly the regularized Thirring model prediction to the energy eigenvalues of the XXZ spin chain model, we start the equation for the boson in the regularized Thirring model where we still keep the box length L finite. The equation for the boson wave function becomes just the same as the massive Thirring model [13] and can be written as [5,6]

$$Ef_n = 2E_{p_n}f_n - \frac{g}{L}\sum_{l=-N_0}^{N_0} f_l\left(1 + \frac{M^2}{E_{p_n}E_{p_l}} + \frac{p_n p_l}{E_{p_n}E_{p_l}}\right), \quad (3.4\text{a})$$

where p_n and E_{p_n} are given as

$$p_n = \frac{2\pi}{L}n\,, \qquad E_{p_n} = \sqrt{M^2 + p_n^2}\,. \quad (3.4\text{b})$$

Further, the induced fermion mass M is given as the solution of the following equation:

$$\frac{g}{L}\sum_{n=-N_0}^{N_0}\frac{1}{\sqrt{M^2 + p_n^2}} = 1\,. \quad (3.5)$$

N_0 is related to the cutoff momentum Λ by

$$\Lambda = \frac{2\pi}{L}N_0\,. \quad (3.6)$$

In order to connect the present calculation with the spin chain, we write the box length L in terms of the lattice spacing constant a as

$$L = Na\,, \quad (3.7)$$

with $N = 2N_0 + 1$. Thus, for large N, we obtain

$$\Lambda = \frac{\pi}{a}\,. \quad (3.8)$$

Equation (3.3) can be easily solved by defining A and B as [13,22]

$$A = \sum_{n=-N_0}^{N_0} f_n\,, \quad (3.9\text{a})$$

$$B = \sum_{n=-N_0}^{N_0}\frac{f_n}{E_n}\,. \quad (3.9\text{b})$$

In this case, we obtain f_n as

$$f_n = \frac{g}{L}\frac{A + \frac{M^2}{E_n}B}{2E_n - E}\,. \quad (3.10)$$

Putting f_n into (3.9), we obtain the eigenvalue equation for E, and this can be solved in a straightforward manner.

4 Bethe ansatz solutions of the Thirring model

The Thirring model is solved by the Bethe ansatz technique [14]. The symmetric solution of the vacuum state has been known and is considered to be the real vacuum state [21,23]. However, we find the new symmetry broken vacuum state in the Bethe ansatz solutions of the Thirring model. This new vacuum energy is lower than the symmetric vacuum energy. Detailed discussions will be found in [15].

From the Bethe ansatz solutions, one obtains the equations of the periodic boundary conditions for the vacuum state [14,19,23,24]:

$$k_i = \frac{2\pi n_i}{L} - \frac{2}{L}\sum_{j\neq i}^{N}\tan^{-1}\left[\frac{g}{2}S_{ij}\right], \quad (4.1\text{a})$$

$$S_{ij} = \frac{k_i|k_j| - k_j|k_i|}{k_ik_j - |k_i||k_j| - \epsilon^2}\,, \quad (4.1\text{b})$$

where S_{ij} denotes the phase shift function. Here, ϵ denotes the infrared regulator which should be infinitesimally small. k_i denotes the momentum of the ith particle. The n_i are integer, and run over $n_i = 0, \pm 1, \pm 2, \ldots, N_0$, with $N_0 = (N-1)/2$. Here, N and L denotes the particle number and the box length, respectively. The cutoff momentum Λ is related to the N and L by

$$\Lambda = \frac{2\pi N_0}{L}\,.$$

For the infrared regulator ϵ, it is important to note that the physical observables, like the momentum k_i, do not depend on the regulator ϵ. Further, the symmetric solutions of (4.1) are just the same as those given by the calculations of the other methods. The derivation of (4.1) is given in the appendix.

The symmetric solution of (4.1) is known and is written as [23]

$$k_1 = 0\,, \tag{4.2a}$$

for $n_1 = 0$,

$$k_i = \frac{2\pi n_i}{L} + \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right), \tag{4.2b}$$

for $n_i = 1, 2, \ldots, N_0$, and

$$k_i = \frac{2\pi n_i}{L} - \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right), \tag{4.2c}$$

for $n_i = -1, -2, \ldots, N_0$. In this case, the vacuum energy in units of the cutoff Λ is given by

$$\mathcal{E}_{\mathrm{v}} = -\sum_{i=1}^{N} |k_i|/\Lambda\,. \tag{4.3}$$

From (4.2), we obtain the symmetric vacuum energy, and it was considered to be the lowest state.

The symmetric vacuum energy $\mathcal{E}_{\mathrm{v}}^{\mathrm{sym}}$ can be written as

$$\mathcal{E}_{\mathrm{v}}^{\mathrm{sym}} = -\left[N_0 + 1 + \frac{2N_0}{\pi}\tan^{-1}\left(\frac{g}{2}\right)\right]. \tag{4.4}$$

Now, we wish to discuss new solutions in (4.1). Here, we find that, in (4.1), there is a symmetry broken vacuum state which is lower than the above symmetric vacuum. By the numerical calculation of (4.1), we first find the symmetry broken vacuum state. Now, we call it true vacuum. After that, we get to know that the solutions can be analytically written like the symmetric case,

$$k_1 = \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right), \tag{4.5a}$$

for $n_1 = 0$,

$$k_i = \frac{2\pi n_i}{L} + \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right), \tag{4.5b}$$

for $n_i = 1, 2, \ldots, N_0$, and

$$k_i = \frac{2\pi n_i}{L} - \frac{2(N_0+1)}{L}\tan^{-1}\left(\frac{g}{2}\right), \tag{4.5c}$$

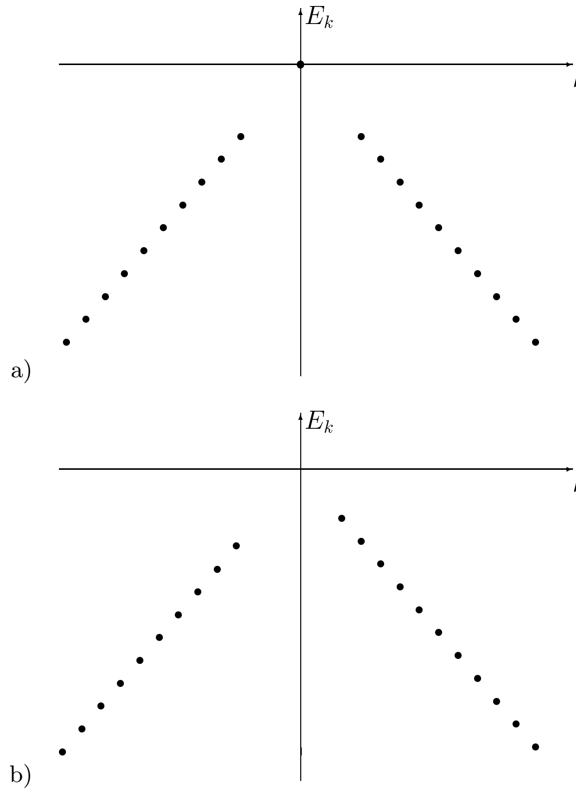

Fig. 1. We show the momentum distribution of the symmetry preserved vacuum **a**, and the true vacuum **b** for $g = 0.5\pi$ with the particle number $N = 21$. **a** There is the zero mode ($k = 0$) in the symmetry preserved vacuum. **b** There is no zero mode in the true vacuum

for $n_i = -1, -2, \ldots, -N_0$.

The true vacuum has no $k_i = 0$ solution, and, instead, all of the momenta of the negative energy particles become finite. In Fig. 1, we show the momentum distribution of symmetric and true vacuum states. Note that the true vacuum state is degenerate due to the $k \leftrightarrow -k$ symmetry, which is always the case with Bethe ansatz solutions. There is no zero mode ($k = 0$) in the true vacuum while there exists a zero mode in the symmetric vacuum. The energy of the true vacuum state $\mathcal{E}_{\mathrm{v}}^{\mathrm{true}}$ can be written as

$$\mathcal{E}_{\mathrm{v}}^{\mathrm{true}} = -\left[N_0 + 1 + \frac{2(N_0+1)}{\pi}\tan^{-1}\left(\frac{g}{2}\right)\right]. \tag{4.6}$$

In Table 1, we show the calculated results of the vacuum energy. As can be seen, the true vacuum state is indeed lower than the symmetric vacuum energy. From the momentum distribution of the negative energy particles, we see that the true vacuum state indeed breaks the chiral symmetry. This situation can be easily seen from the analytical solutions since the absolute value of the momentum of the negative energy particles is higher than $2N_0/L\tan^{-1}(g/2)$. Therefore, we can define the effective fermion mass $\mathcal{M}_N$ by

$$\mathcal{M}_N = \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right). \tag{4.7}$$

Further, we carry out the calculations of the excitation energy of the one particle–one hole ($1p$–$1h$) state. There,

Table 1. The vacuum energy of Bethe ansatz solutions is shown for $g = 0.5\pi$ with the particle number $N = 401$ and $N = 1601$. $\mathcal{E}_{\mathrm{v}}^{\mathrm{sym}}$ and $\mathcal{E}_{\mathrm{v}}^{\mathrm{true}}$ denote the symmetric vacuum and the true vacuum energies, respectively. The effective fermion mass $\mathcal{M}_N$ deduced from the vacuum momentum distributions is also shown. All the energies are in units of Λ

N	$\mathcal{E}_{\mathrm{v}}^{\mathrm{sym}}$	$\mathcal{E}_{\mathrm{v}}^{\mathrm{true}}$	$\mathcal{M}_N$
401	−285.769	−286.193	0.212
1601	−1140.076	−1140.500	0.212

we take out one negative energy particle (i_0th particle) and put it into a positive energy state. In this case, (4.1) become

$$k_i = \frac{2\pi n_i}{L} - \frac{2}{L}\tan^{-1}\left(\frac{g}{2}\tilde{S}_{ii_0}\right) - \frac{2}{L}\sum_{j\neq i,i_0}^{N}\tan^{-1}\left(\frac{g}{2}S_{ij}\right), \tag{4.8a}$$

for $i \neq i_0$, and

$$k_{i_0} = \frac{2\pi n_{i_0}}{L} - \frac{2}{L}\sum_{j\neq i_0}^{N}\tan^{-1}\left(\frac{g}{2}\tilde{S}_{i_0 j}\right), \tag{4.8b}$$

for $i = i_0$.

Here, $\tilde{S}_{i_0 j}$ is given by

$$\tilde{S}_{i_0 j} = \frac{k_{i_0}|k_j| + k_j|k_{i_0}|}{k_{i_0}k_j + |k_{i_0}||k_j| + \epsilon^2}. \tag{4.8c}$$

In this case, the energy of the one particle–one hole states $E_{(i_0)}^{1p1h}$ is given by

$$E_{(i_0)}^{1p1h} = |k_{i_0}| - \sum_{\substack{i=1\\ i\neq i_0}}^{N}|k_i|. \tag{4.9}$$

It turns out that the solutions of (4.8) can be found at the specific value of n_{i_0} and then from this n_{i_0} value on, we find the continuous spectrum of the $1p$–$1h$ states.

Here, we show the analytical solution of (4.8) for the first $1p$–$1h$ state.

$$k_{i_0} = \frac{2\pi n_{i_0}}{L} - \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right), \tag{4.10a}$$

for n_{i_0},

$$k_i = \frac{2\pi n_i}{L} + \frac{2(N_0+1)}{L}\tan^{-1}\left(\frac{g}{2}\right), \tag{4.10b}$$

for $n_i = 0, 1, 2, \ldots, N_0$, and

$$k_i = \frac{2\pi n_i}{L} - \frac{2N_0}{L}\tan^{-1}\left(\frac{g}{2}\right), \tag{4.10c}$$

for $n_i = -1, -2, \ldots, -N_0$.

Here, n_{i_0} is given by

$$n_{i_0} = \left[\frac{N_0}{\pi}\tan^{-1}\left(\frac{g}{2}\right)\right], \tag{4.11}$$

Table 2. Several lowest states of the calculated $1p$–$1h$ states energy are shown at $g = 0.5\pi$ with $N = 1601$. The gap energy $\Delta\mathcal{E} \equiv \mathcal{E}_0^{1p1h} - \mathcal{E}_{\mathrm{v}}$ is also shown. All the energies are measured in units of Λ

	$\mathcal{E}$	$\Delta\mathcal{E}$
Vacuum	−1140.500	
$1p$–$1h$ (1)	−1140.075	0.425
$1p$–$1h$ (2)	−1140.074	0.426
$1p$–$1h$ (3)	−1140.072	0.428
$1p$–$1h$ (4)	−1140.071	0.429
$1p$–$1h$ (5)	−1140.070	0.430
$1p$–$1h$ (6)	−1140.069	0.431

where $[X]$ denotes the smallest integer value which is larger than X. In this case, we can express the lowest $1p$–$1h$ state energy in units of the cutoff Λ analytically

$$\mathcal{E}_0^{1p1h} = -\left[(N_0+1) - \frac{2n_{i_0}}{N_0} + \frac{2(N_0+1)}{\pi}\tan^{-1}\left(\frac{g}{\pi}\right)\right]. \tag{4.12}$$

Therefore, the lowest excitation energy $\Delta\mathcal{E}_0^{1p1h}$ with respect to the true vacuum state becomes

$$\Delta\mathcal{E}_0^{1p-1h} \equiv \mathcal{E}_0^{1p1h} - \mathcal{E}_{\mathrm{v}}^{\mathrm{true}} = \frac{2n_{i_0}}{N_0}. \tag{4.13}$$

If we take the thermodynamic limit, that is, $N \to \infty$ and $L \to \infty$, then (4.12) can be reduced to

$$\Delta\mathcal{E}_0^{1p1h} = \frac{2}{\pi}\tan^{-1}\left(\frac{g}{2}\right) = 2\mathcal{M}_N/\Lambda. \tag{4.14}$$

From this gap energy, we can obtain the effective fermion mass which is one half of the lowest gap energy. It turns out that the fermion mass from the gap energy is consistent with the one deduced from the momentum distribution of the vacuum state in Table 1.

In Table 2, we show the several lowest states of the $1p$–$1h$ energy by numerical calculation. As can be seen, there is a finite gap in the excitation spectrum. This calculation of the Thirring model in terms of the Bethe ansatz solutions confirms the calculated results of the Bogoliubov transformation method for the symmetry breaking phenomena.

Here, we comment on the bosonic excitation spectrum which is predicted by the Bogoliubov transformation method of the Thirring model. The massless Thirring model can be exactly solved by the Bethe ansatz method. In this formulation we show that there is no bosonic excitation in the massless Thirring model. This indicates that the Bogoliubov transformation method of the Thirring model is not exact. In Fig. 2, we show the dispersion relation of the symmetry broken vacuum predicted by the Bethe ansatz, and it can be fit by the function

$$E_k = -\sqrt{k^2 + \xi^2}, \tag{4.15}$$

where ξ^2 is a constant. Therefore, the Bogoliubov transformation method may present a good approximate scheme for the analysis of the Thirring model. In particular, the

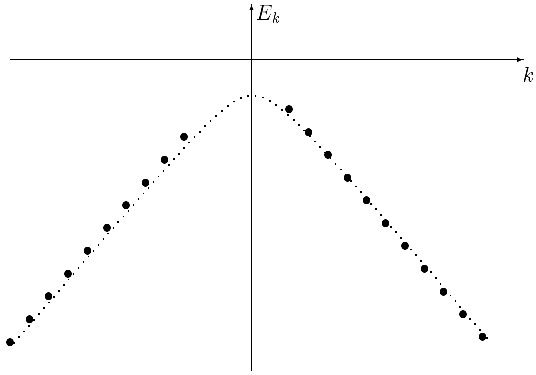

Fig. 2. The bullets (•) show the Bethe ansatz solutions for the true vacuum of the Thirring model. The dot line shows the dispersion relation ($E_k = -\sqrt{k^2+\xi^2}$, where ξ^2 is a constant) which is based on the Bogoliubov transformation calculation

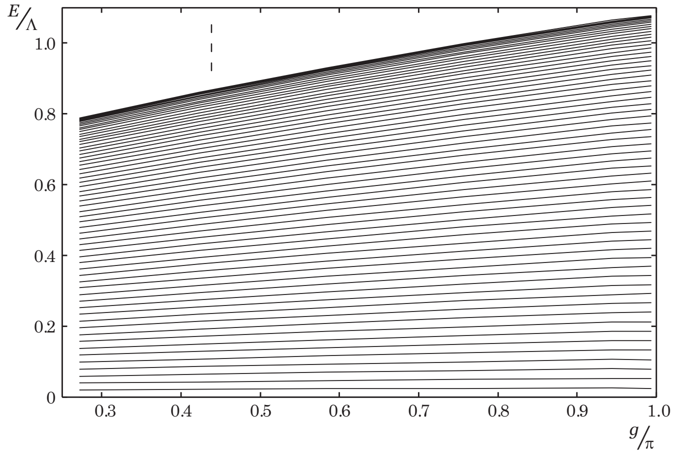

Fig. 3. The calculated spectrum of the XXZ spin chain

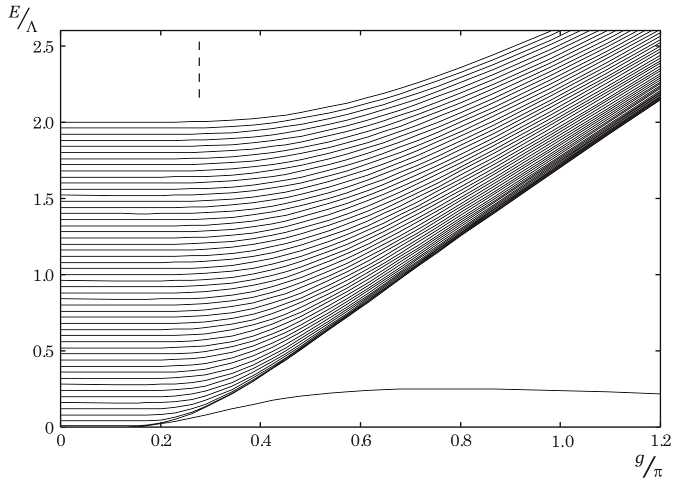

Fig. 4. The predicted spectrum of the regularized Thirring model

vacuum property may be described well by the Bogoliubov transformation method. However, the Bogoliubov transformation tends to overestimate the attraction between fermions and antifermions since it predicts a bosonic bound state.

5 Non-equivalence between Heisenberg XXZ and Thirring model

The Heisenberg XYZ spin chain is known to be equivalent to the massive Thirring model at the continuum limit [9]. The translation of the coupling constants between XYZ and massive Thirring models is given in (2.7).

From the above equivalence between the XYZ spin chain and the massive Thirring models, one also expects the equivalence between the XXZ and the massless Thirring models since the XXZ spin chain corresponds to the massless limit of the XYZ spin chain. However, the massless limit is a singular point in the massive Thirring model, and therefore it is non-trivial whether the XXZ spin chain and the massless Thirring model are equivalent to each other. In particular, the massless Thirring model has no scale, and therefore, one has to introduce the cutoff momentum Λ by which all of the observables must be measured. On the other hand, the XXZ spin chain has a natural scale of the lattice constant, and this is an important contrast to the massless Thirring model.

In Figs. 3 and 4, we show the excitation spectrum of the two models. Figure 3 shows the calculated results of the XXZ spin chain by solving the Bethe ansatz equations while Fig. 4 shows the predicted spectrum of the regularized Thirring model calculated by the Bogoliubov transformation method.

As can be seen from these figures, there is a significant difference between them. In the XXZ spin chain, there is no gap in the first excited state, but the regularized Thirring model has a finite gap in the first excited state. This finite gap is also confirmed by the Bethe ansatz solutions of the massless Thirring model.

This means that the two models are not equivalent to each other, even though it is believed that the XXZ spin chain at the continuum limit corresponds to the massless Thirring model [10] if one considers the excitations near the Fermi sea.

What is wrong with the derivation of the Thirring model from (2.9)? Here, we present our interpretation of the non-equivalence of the two models. In the XYZ spin chain, one can make a continuum limit since there are two parameters which have dimensions, the lattice constant and the mass parameter. Therefore, one can make the proper continuum limit in the XYZ spin chain. However, when one makes a massless limit from the XYZ to XXZ, then the XXZ possesses only one scale, the lattice constant. In this case, one cannot take the continuum limit since everything is already measured by the lattice constant. The equivalence between the XXZ spin chain and the massless Thirring model derived up to now must be due to the improper procedure of the continuum limit in the XXZ spin chain. In this respect, one should say that there is no corresponding field theory of the XXZ spin chain in the continuum limit, and therefore it does not correspond to the massless Thirring model.

In fact, this continuum field theory of the massless Thirring model possesses the chiral symmetry which is not shared by the XXZ spin chain. This continuous symmetry plays a very important role for the vacuum structure. In the massless Thirring model, there are two vacua, one which preserves the chiral symmetry, and the other which violates the chiral symmetry. Under the chiral symmetry breaking, the true vacuum goes to the one which is lower than the trivial vacuum. In this respect, the physical vacuum of the massless Thirring model is the one that violates the chiral symmetry. The spectrum should be constructed on this physical vacuum state. On the other hand, the XXZ spin chain does not possess this chiral symmetry. and therefore naturally the spectrum of the XXZ spin chain corresponds to the one obtained from the trivial vacuum state of the Thirring model.

6 Conclusions

We have examined the relation between the spin 1/2 Heisenberg XXZ model and the massless Thirring model. It turns out that the spectrum of the XXZ model is different from the massless Thirring model, and it does not possess the true vacuum of the massless Thirring model. In this respect, the equivalence between the XXZ and the Thirring models does not hold, contrary to the case of the massive theory in which the XYZ spin chain is indeed equivalent to the massive Thirring model at the continuum limit.

This is essentially due to the fact that the massless Thirring model has the continuous symmetry (chiral symmetry) while the XXZ spin chain does not possess such a symmetry. Therefore, in the massless Thirring model, there are two vacua, one which keeps the chiral symmetry, and the other which violates the chiral symmetry. The true vacuum is the one that violates the chiral symmetry since it is lower than the other. But the XXZ spin chain cannot reproduce the true vacuum state of the massless Thirring model.

This may indicate rather an important consequence concerning the lattice version of the continuous field theory model. Once the lattice version of the field theory model loses some important continuous symmetry, then the lattice version may not be able to reproduce some physically important spectrum of the continuous field theory model. Although the XXZ spin chain has of course its own interest in physics, it should not serve as the lattice version of the continuous field theory model.

A Derivation of (4.1)

We present the derivation of (4.1) from the Bethe ansatz equations of the massive Thirring model which are given by Bergknoff and Thacker [14]. According to the Bethe ansatz, the Hamiltonian can be digonalized when the phase shift function S_{ij} is written as

$$S_{ij} = \frac{\sin(\theta_{k_i} - \theta_{k_j})}{\sin(\theta_{k_i} + \theta_{k_j})}\,, \tag{A.1}$$

where

$$\tan 2\theta_{k_i} = \frac{m_0}{k_i}\,. \tag{A.2}$$

Therefore, we can rewrite it as

$$\sin\theta_{k_i} = \sqrt{\frac{1-\cos 2\theta_{k_i}}{2}} = \frac{\sqrt{E_i - k_i}}{\sqrt{2E_i}}\,, \tag{A.3}$$

$$\cos\theta_{k_i} = \sqrt{\frac{1+\cos 2\theta_{k_i}}{2}} = \frac{\sqrt{E_i + k_i}}{\sqrt{2E_i}}\,, \tag{A.4}$$

where $E_i = \sqrt{k_i^2 + m_0^2}$. Thus, we have

$$\sin(\theta_{k_i} - \theta_{k_j}) = \frac{1}{2\sqrt{E_iE_j}} \times \left[\sqrt{E_i - k_i}\sqrt{E_j + k_j} - \sqrt{E_i + k_i}\sqrt{E_j - k_j}\right], \tag{A.5a}$$

$$\cos(\theta_{k_i} - \theta_{k_j}) = \frac{1}{2\sqrt{E_iE_j}} \times \left[\sqrt{E_i - k_i}\sqrt{E_j + k_j} + \sqrt{E_i + k_i}\sqrt{E_j - k_j}\right]. \tag{A.5b}$$

In this case, S_{ij} becomes

$$S_{ij} = \frac{\sqrt{E_i - k_i}\sqrt{E_j + k_j} - \sqrt{E_i + k_i}\sqrt{E_j - k_j}}{\sqrt{E_i - k_i}\sqrt{E_j + k_j} + \sqrt{E_i + k_i}\sqrt{E_j - k_j}} = \frac{k_iE_j - k_jE_i}{k_ik_j - E_iE_j - m_0^2}\,. \tag{A.6}$$

For the massless limit $m_0 \to 0$, $E_i = \sqrt{k_i^2 + m_0^2} \to |k_i|$. Therefore, we have the phase shift function S_{ij} of the *massless* Thirring model with the *regulator* ϵ as follows:

$$S_{ij} = \frac{k_i|k_j| - k_j|k_i|}{k_ik_j - |k_i||k_j| - \epsilon^2}\,. \tag{A.7}$$

Here, it should be important to note that the solutions of (4.1) do *not* depend on the regulator ϵ. Therefore, we can take the massless limit properly.

References

1. J. Goldstone, Nuovo Cimento **19**, 154 (1961)
2. J. Goldstone, A. Salam, S. Weinberg, Phys. Rev. **127**, 965 (1962)
3. S. Coleman, Comm. Math. Phys. **31**, 259 (1973)
4. Y. Nambu, G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961)
5. M. Hiramoto, T. Fujita, No massless boson in chiral symmetry breaking in NJL and Thirring model, hep-th/0306083
6. T. Fujita, M. Hiramoto, H. Takahashi, No Goldstone boson in NJL and Thirring model, hep-th/0306110
7. T. Fujita, M. Hiramoto, H. Takahashi, Historic mistake in spontaneous symmetry breaking, hep-th/0410220
8. M. Faber, A.N. Ivanov, Eur. Phys. J. C **20**, 723 (2001)
9. A. Luther, Phys. Rev. B **14**, 2153 (1976)
10. I. Affleck, Les Houches, Session XLIX, 1990, North-Holland
11. H.A. Bethe, Z. Phys. **71**, 205 (1931)
12. R. Orbach, Phys. Rev. **112**, 309 (1958)

13. M. Hiramoto, T. Fujita, Phys. Rev. D **66**, 045007 (2002)
14. H. Bergknoff, H.B. Thacker, Phys. Rev. Lett. **42**, 135 (1979)
15. T. Fujita, M. Hiramoto, T. Homma, H. Takahashi New Vacuum of Bethe Ansatz Solutions in Thirring Model, J. Phys. Soc. Jpn., in press, hep-th/0410221
16. T. Tomachi, T. Fujita, Ann. Phys. **223**, 197 (1993)
17. T. Fujita, M. Hiramoto, T. Homma, New spectrum and condensate in two dimensional QCD, Prog. Theor. Phys. in press, hep-th/0306085
18. T. Fujita, T. Kobayashi, H. Takahashi, Phys. Rev. D **68** , 068701 (2003)
19. T. Fujita, Y. Sekiguchi, K. Yamamoto, Ann. Phys. **255**, 204 (1997)
20. T. Fujita, T. Kobayashi, H. Takahashi, J. Phys. A **36**, 1553 (2003)
21. W. Thirring, Ann. Phys. (N.Y) **3**, 91 (1958)
22. T. Fujita, A. Ogura, Prog. Theor. Phys. **89**, 23 (1993)
23. K. Odaka, S. Tokitake, J. Phys. Soc. Jpn. **56**, 3062 (1987)
24. N. Andrei, J.H. Lowenstein, Phys. Rev. Lett. **43**, 1698 (1979)